A High-Precision Curved Shell Finite Element

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Theme

A REFINED finite element procedure applicable to the static analysis of arbitrarily loaded, thin elastic shells of revolution is developed based on a thin shell theory including the effect of transverse shear deformations.

Contents

The geometry of the finite element, shown in Fig. 1, is treated by representing various combinations of the definitive geometrical parameters with fourth-order Lagrangian interpolation polynomials. The coefficients of the polynomials are evaluated from exact geometrical data. These polynomials, P_k, are expressed in terms of a nondimensionalized arc length variable s and take the form

$$P_k = \lfloor 1 s s^2 s^3 s^4 \rfloor \{\bar{P}_{km}\} \tag{1}$$

in which

$$\{\bar{P}_{km}\} = [C]\{P_{k0} \dots P_{km} \dots P_{k4}\} \qquad (k = 1, 14) \tag{2}$$

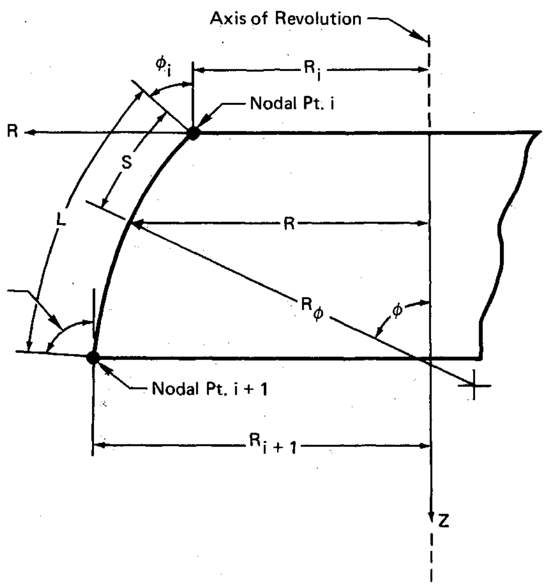

Fig. 1 Finite element geometry.

P_{km} = the known value of P_k at the quarter points measured along the meridian, and $[C]$ = a matrix of constants. Approximating these geometrical functions in this manner allows numerous integrals to be evaluated in closed form and avoids differentiation of interpolation polynomials.

The displacement functions are taken in the harmonic form

$$D = \{u v w \beta_\phi \beta_\theta\} = \sum_{j=0}^{\infty} \{u^{(j)} \cos j\theta \;\; v^{(j)} \sin j\theta \;\; w^{(j)} \cos j\theta \;\; \beta_\phi{}^{(j)} \cos j\theta \;\; \beta_\theta{}^{(j)} \sin j\theta\} \tag{3}$$

in which u, v and w = the meridional, circumferential and normal displacements and β_ϕ and β_θ = the meridional and circumferential rotations.

Each displacement function is represented as an interpolation polynominal of the form

$$\bar{d}_i(s) = (1-s)\, d_i(0) + s d_i(1) + (s-s^2)\, q_i{}^1 + \dots + (s^{\xi_i} - s^{\xi_i+1})\, q_{i\xi_i} \tag{4}$$

in which $\bar{d}_i(s)$ = the s-dependent part of the ith element of vector $\{D\}$; $d_i(0), d_i(1)$ = nodal values of generalized displacement $\bar{d}_i(s)$ and also coefficients of linear terms in the displacement functions, and $q_{i1} \dots q_{i\xi_i}$ = coefficients of higher-order terms in the displacement functions. It should be noted that the higher order terms vanish at the nodal circles; however, they serve to provide an improved representation between nodes.

Once the form of the displacement functions has been established, the potential energy functional is determined by well-known procedures. Employing the Principle of Minimum Total Potential Energy and the static condensation reduction, the nodal equilibrium equations are

$$[K]\{\Delta\} = \{F\} \tag{5}$$

in which $[K]$ = the refined element stiffness matrix, $\{\Delta\}$ = the nodal displacement vector, and $\{F\}$ = the refined element nodal loading vector.

For the solution of closed shells of revolution, suitable cap elements are derived predicated on the assumption of finite strains at the pole, from which the required displacement conditions are established.

Improving the element stiffness matrices and loading vectors

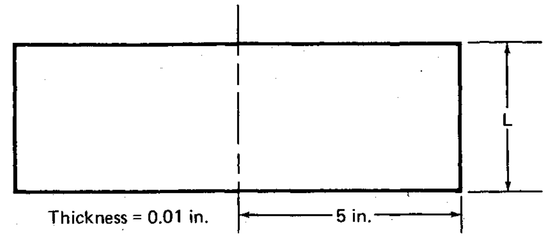

Fig. 2 Cylindrical shell element.

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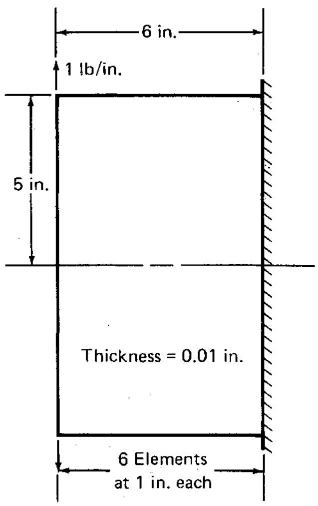

Fig. 3 Cylindrical shell—edge loading.

allows the number of elements required to obtain solutions of acceptable accuracy to be significantly reduced. Intermediate stress resultants are determined by substituting the displacement polynomials into appropriate stress resultant-displacement relations. Since the proposed technique employs high-order displacement functions, the derivatives of these functions are sufficiently accurate to calculate the internodal stress resultants.

The most comprehensive basis of comparison of individual finite elements having the same geometry and degrees of freedom is provided by the eigenvalues of the stiffness matrix; the trace of an element stiffness matrix, which is the sum of the diagonal terms and equal to the sum of the eigenvalues, provides a convenient measure of the quality of the element. The lower the trace, the more flexible and therefore the more satisfactory the element will be. The length of a cylindrical element, shown in Fig. 2 was allowed to vary and the traces compared as higher-order displacement terms were added. The results are presented in Table 1 where the convergence is shown to be quite rapid.

A cylindrical shell subject to a concentrated line load of 1 lb/in. as shown in Fig. 3 is analyzed. This problem is often cited in comparative studies presented in the literature. Large elements are defined so that the effect of adding additional terms to each displacement function can be demonstrated. The meridional moment M_ϕ is plotted in Fig. 4. It is apparent that the computed moment curve approaches the actual moment curve as more terms are added to the displacement functions. The further division of the first element into two elements $\frac{1}{2}$ in. in length produces results which agree within 0.5% with the theoretical results as indicated by the circles in Fig. 4. Previous published solutions require many more degrees of freedom to achieve comparable accuracy.

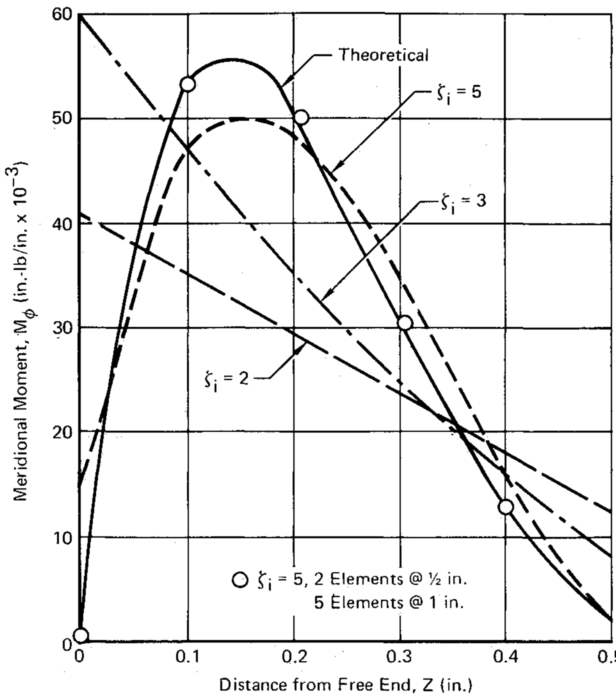

Fig. 4 Meridional moment—edge loaded shell.

The results of several other example problems, including shells with curved meridians, closed ends and asymmetrical loadings, indicate that very good accuracy can be achieved utilizing relatively few degrees of freedom as compared with other works where improved representation of the displacement state between nodes is not employed.

Table 1 Trace of stiffness matrix—cylindrical element

Element length	ζ_i for each displacement function			
	0	1	3	5
0.0625	22.7183	18.7304	17.9975	17.9975
0.1250	11.3705	9.2604	8.8471	8.8471
0.2500	5.7080	4.6209	4.3964	4.3964
0.5000	2.8996	2.3186	2.1528	2.1526
1.0000	1.5409	1.1792	1.0436	1.0394
2.0000	0.9527	0.6312	0.5303	0.5168
3.0000	0.8376	0.4699	0.3670	0.3495